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THE EQUIVALENCE BETWEEN THE CONNECTION AND THE LOOP REPRESENTATION OF QUANTUM GRAVITY

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The recent developments of the "connection" and "loop" representations have given the possibility to show that the two representations are equivalent and that it is possible to transform any result from one representation into the other. The glue between the two representations is the loop transform. Its use, combined with Penrose's binor calculus, gives the possibility of establishing the exact correspondence between operators and states in the connection representation and those in the loop representation.
The main ingredients in the proof of the equivalence are: the concept of embedded spin network, the Penrose graphical method of $SU(2)$ calculus, and the existence of a generalized measure on the space of connections.

The *loop*[2] and the *connection*[1] representations approch to the canonical quantizion of GR (Einstein's general relativity) have the goal of constructing a quantum theory, based on connection $A_a^i(x)$ modulo gauge transformations (in the case of GR the Ashtekar-Sen $SU(2)$ connection), such that the Wilson's loop functionals (the trace of the Holonomy along a closed loop α)

$$\mathcal{T}_\alpha[A] = \mathrm{Tr}\mathcal{P}\exp\Big[\int_\alpha A_a(x)dx^a\Big] \tag{1}$$

become well defined quantum operators. In the *loop-representation* approch the quatization is achieved realizing the *quantum operator* that correspond to the $\mathcal{T}$ observables on the vector space $\mathcal{V}_{loop}$ of all the loops modulo the Mandelstam relations.[a] In the *connection-representation* approch, in contrast, the first step was the construction of the Hilbert space structure in which the $\mathcal{T}_\alpha[A]$ operators are realized as multiplications: the Hilbert space $\mathcal{H} = L^2[\overline{\mathcal{A}/\mathcal{G}}, d\mu]$ of the square integrable function with respect to the Gel'fand spectral measure associated to the $C^\star$ algebra of the $\mathcal{T}_\alpha[A]$'s. These two approaches are connected by the so-called loop trasformation. To any state $\psi_C \in \mathcal{H}$, it is assocated a state $\psi_L \in \mathcal{V}_{loop}$ as:

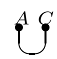
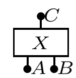
$$\psi_L(\alpha) =<\alpha|\psi>= \int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu(A) <\alpha|A><A|\psi>= \int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu(A)\overline{\mathcal{T}_\alpha[A]}\psi_C(A) \quad . \tag{2}$$

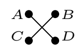
The problem of proving the equivalence of the two representations is equivalent to the problem of showing the explicit action of this transformation. The different mathematical framework of the two representations was the only reason behind the difficulty.

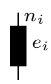
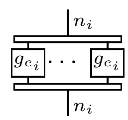
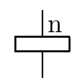
[a] It is important to note that in this approch the essential problem of the definition of the scalar product and indeed of the the Hilbert Space structure was postpone (see [6] for the solution of this problem on the original phylosophy of the *loop-representation*).

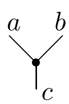
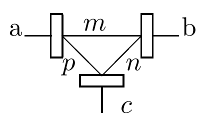
This problem has a straightforward solution using Penrose's graphical binor calculus for the $SU(2)$-tensors in the connection representation[4]. Using this method, it is immediate to show that the loop transform (2) maps the spin-network basis of $\mathcal{V}_{loop}$[5] into the spin-network basis of $\mathcal{H} = L^2[\overline{\mathcal{A}/\mathcal{G}}, d\mu]$ [3]. We refer the interest reader to [4] for a detailed account of the proof and for the relevant bibliography.

The basic idea behind Penrose's binor calculs is to represent any $SU(2)$ tensor (i.e., tensor expression with indices $A, B, \ldots = 1, 2$) in terms of the following graphical elements in the plane:

$\delta_C{}^A =$  $\mathrm{i}\epsilon_{AC} =$  $\mathrm{i}\epsilon^{AC} =$  $X^C_{AB} =$  (3)

and assigns to any crossing a minus sign, i.e: $X^{AB}_{CD} = \delta_D{}^A \delta_C{}^B = -$  . Using this rule it is possible to represent any $SU(2)$ ($SL(2,C)$) tensor expression in a graphical way. In particular we have the following graphical representation for (i) the irreducible representation $\pi_i(n_i)$[b], and of the unique 3-valent contractor

$\pi_i(n_i) =$  $=$  , $\quad \Pi^{(e)}_n P_n = \frac{1}{n!}\sum_p (-1)^{|p|}\; P^{(p)}_p =$  ,

 $\stackrel{def}{=}$  , $\quad \left\{ \begin{array}{ccc} m &=& (a+b-c)/2 \\ n &=& (b+c-a)/2 \\ p &=& (c+a-b)/2 \end{array} \right.$. (4)

Now, the space $\mathcal{H} = L^2[\overline{\mathcal{A}/\mathcal{G}}, d\mu]$ and its spin-network basis are defined as follows: (**i**) the quantum configuration space $\overline{\mathcal{A}/\mathcal{G}}$ is taken to be the Gel'fand spectrum generated by the Wilson loop functionals; (**ii**) the space $\overline{\mathcal{A}/\mathcal{G}}$ could be characterized as the projective limit of the finite dimensional spaces $\overline{\mathcal{A}/\mathcal{G}}_\gamma$ of the cylindrical functions associated to piecewise analytical graphs γ and in this space a fiducial measure $d\mu_0(A)$ is naturally defined as the σ-additive extension of the family of products of Haar measures $d\mu_{0,\gamma}(A) = d\mu_H(g_{e_1})\ldots d\mu_H(g_{e_n})$ in the spaces $\overline{\mathcal{A}/\mathcal{G}}_\gamma$; A function f_γ ($f_\gamma \in \overline{\mathcal{A}/\mathcal{G}}_\gamma$) is said to be cylindrical with respect to a graph γ if it is a gauge invariant function of the finite set of arguments $(g_{e_1}(A), \ldots, g_{e_n}(A))$ where the $g_{e_i} = \mathcal{P}\exp(-\int_{e_i} A)$ are the holonomies of A along the edges e_i of the graph γ. (**iii**) a natural basis in the space $\overline{\mathcal{A}/\mathcal{G}}$ is given by the spin-network cylindrical functions. They express the fact that any function cylindrical with respect to a graph γ can be decomposed in terms of irreducible representations, i.e.

$$f_\gamma(g_{e_1}, \ldots, g_{e_n}) = \sum_{\vec{\pi},\vec{c}} f(\gamma, \vec{\pi}, \vec{c})\;\; \mathcal{T}_{\gamma,\vec{\pi},\vec{c}}[A],\;\; \mathcal{T}_{\gamma,\vec{\pi},\vec{c}}[A] \stackrel{def}{=} \left(\otimes^{\#^{\mathrm{edge}}}_{i=1} \pi_i(g_{e_i})\right) \cdot \left(\otimes^{\#^{\mathrm{vetex}}}_{j=1} c_j\right)$$

where: (**i**) $\vec{\pi} = (\pi_1, \ldots, \pi_N)$ denotes the labeling of the edges with irreducible representation π_i of G; (**ii**) $\vec{c} = (c_1, \ldots, c_M)$ a labeling of the vertices with invariant

[b] Labeled by an integer n, its color, that is twice the spin: $n = 2j_n$

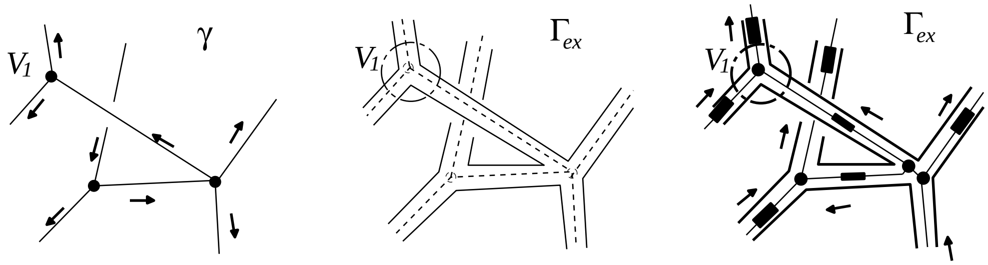


Figure 1: The graph γ, a possible extended-planar projection Γ_{ex} and the graphical representation of a spin-network cilyndrical function

contractors c_j (the intertwining matrices c_j, in each of the vertices v_j, represent the invarian coupling of the n_j representations associated to the n_j edges that start or end in v_j).

At this point, it is immediate to define the graphical binor representation of the spin network $\mathcal{T}_{\gamma,\vec{\pi},\vec{c}}[A]$. Refering to Fig.1, (1) cosider a planar projection of the graph γ and (2) its extended planar projection Γ_{ex}; (3) insert in any extended-edge the graph-rep. of the irreduciple representation and in any extended-vertex the graph-rep. of the contractor in terms of its decomposition in terms of 3-valent invariant tensor; (4) represent index contraction as the joining of the corresponding lines.

Now, consider the definition of the spin network state in the loop representation given in [6]. We are left with the task of proving that the loop-transform of them is exactly a spin-network state of the connection representation. Refering to eq. (2) we have to show $< A, \alpha >_{Loop} = T_{\gamma,\vec{\pi},\vec{c}}[A]$. Recalling that a spin network in the loop representation (section V of [6]) is exactly the drawing on Γ_{ex} corresponting to the graphical binor-representation of a spin-network basis of the connection representation (in the normalization discused in the previous section), the assertion that the Loop-Transform of a spin-network of the loop representation is a spin-network of the connection representation follow in a straightforward way.

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